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Metric of the multiply wound rotating string

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Abstract

We consider a string wrapped many times around a compact circle in space, and let this string carry a right moving wave which imparts momentum and angular momentum to the string. The angular momentum causes the strands of the ‘multiwound’ string to separate and cover the surface of a torus. We compute the supergravity solution for this string configuration. We map this solution by dualities to the D1-D5 system with angular momentum that has been recently studied. We discuss how constructing this multiwound string solution may help us to relate the microscopic and macroscopic pictures of black hole absorption.

1 Introduction.

String theory has been remarkably successful in explaining the thermodynamic properties of black holes in terms of the statistical mechanics of a unitary microscopic system. For example one can make a black hole in 4+1 dimensions by compactifying 5 directions of 10-d Minkowski space, and then wrapping D5 and D1 branes around these compact directions [1]. In the microscopic description of the D1-D5 system at weak coupling, an incoming graviton is absorbed when its energy gets converted to the energy of vibrations traveling along compact direction where the D1 brane is wrapped [2]. Even though the size of this compact direction is small (say Planck length), very low energy quanta can be absorbed by the ‘effective string’ formed by the D1 and D5 branes due to the possibility of ‘multiwinding’ [3]: if a string is wrapped n times along a circle of length $2\pi R$ then it can carry excitations of wavelength $2\pi Rn$ rather than $2\pi R$, and thus the excitation threshold of such a string is low if n is large. It was shown in [4] that this excitation threshold in fact agrees with the energy of the last few quanta that are emitted thermally from a near extremal hole; these quanta have a wavelength much larger than the radius of the hole (which is itself a macroscopic length).

On the other hand in the ‘macroscopic picture’ of absorption, a low energy quantum travels down the throat of the 4+1 dimensional metric and falls through the horizon. We do not see any sign of ‘multiwinding’ around the compact direction, and in fact do not see the energy of the quantum being converted to energy in the compact directions. The smallness of the energy threshold for absorption just stems from the macroscopic diameter of the hole and the long length of the throat. To better understand the fate of quanta falling into a black hole and to obtain a resolution of the ‘information paradox’ we need to find a closer link between these two descriptions of the absorption process.

In this paper we study a string which carries momentum, but also carries a large amount of angular momentum. This angular momentum causes the strands of the ‘multiwound string’ to spread out over a surface of macroscopic size, and we study the spacetime metric produced by such a configuration. It is hoped that this study will eventually help us to understand better the relation between the microscopic and macroscopic pictures of absorption by a black hole.

1.1 Angular momentum and the multiwound string

Consider Type IIB string theory on flat spacetime, and let one spatial direction X^9 be compactified to a circle of length $2\pi R$. Let a D-string be wrapped once around this S^1 . Further, we put momentum excitations moving in one direction along the string, carrying a total momentum

$$P = \frac{n_p}{R} \tag{1.1}$$

How much angular momentum can this system have (in the transverse space X^1, \dots, X^8)? The unexcited D-string could itself be in one of 256 ground states, which have spins ranging from 0 to 2. We will ignore this spin in what follows. The momentum modes

are carried by open strings moving along the D-string. Each open string is in one of the massless ground states, and carries a momentum n_i/R , with $\sum n_i = n_p$. The maximum angular momentum is achieved if we let each open string have $n_i = 1$, and further let it be in one of the bosonic ground states $d_{-1/2}^\mu|0\rangle_{NS}$. Aligning all the vector indices μ gives the state with angular momentum

$$J_{max} = n_p = PR \quad (1.2)$$

What happens if the winding number of the D-string is n_w instead of unity? If we have n_w ‘separately wound’ D-strings, then each open string lives on one of the D-strings, and the value J_{max} remains that given in (1.2). But the n_w D-strings can join up into one long ‘multiply wound’ D-string of total length $2\pi R n_w$. Then the total momentum is still of the form (1.1), but the individual open strings carrying this momentum have

$$p_i = \frac{n_i}{n_w R}, \quad \sum_i n_i = n_w n_p \quad (1.3)$$

The maximum angular momentum is again obtained when all the n_i are unity (and the vector spins are all aligned), but the value of this maximum angular momentum is

$$J_{max} = n_w n_p = n_w PR \quad (1.4)$$

Note that when each of the momentum modes has such a fractional value $1/(n_w R)$ of momentum then the vibration they describe has a wavelength $2\pi n_w R$ along the D-string. In executing such a vibration the n_w different strands of the D-string will have to move apart from each other; they will not move together as a single high density string on the interval $0 \leq X^9 \leq 2\pi R$. In this paper we wish to study the metric produced by a D-string which is in such a microscopic configuration. The solution we obtain will have momentum and angular momentum related as in (1.4). From the above argument such a configuration should necessarily exhibit a spread among the strands of the D-string; this is an effect that we wish to see.

In more detail, we carry out the following computations:

(a) First we look at a classical string described by the Born-Infeld action. If we have a purely right-moving wave on such a string, then we show that the angular momentum carried by the string is bounded by the value (1.4). As a byproduct of this computation we note the geometry of a string carrying the maximal allowed angular momentum: the traveling wave on the string makes the string profile a helix, which turns around a circle $X_1^2 + X_2^2 = a^2$ while moving up in a direction X_9 along which the string is wrapped. With a large value for the winding number n_w we find that there are many strands of the string at any given value of z , so that the string covers in a dense fashion the surface of the torus given by the product of the circle $X_1^2 + X_2^2 = a^2$ and the circle in the direction X_9 .¹

¹A related calculation using 2-branes and 0-branes has been carried out in [5].

(b) We then make a supergravity solution that will describe such a string configuration. This solution will carry momentum, string winding charge and angular momentum. We start with the metric of a neutral black hole carrying angular momentum, and transform it by a sequence of boosts and T-dualities so that it has momentum and winding charges. The location of the string can be found by looking for the points where the dilaton goes to zero: we observe that this hypersurface has the form of the torus mentioned in the paragraph above.

(c) In the above method of generating the supergravity solution we do not explicitly see the strands of the string and how they wind around the torus. We now derive the same supergravity solution by a different technique: we start with the known solution of an oscillating string, take it to have a configuration that exhibits ‘multiwinding’, and then smear this string uniformly over the analog of the circle $X_1^2 + X_2^2 = a^2$ to obtain a distribution that is again uniform on the torus. Having derived the solution this way, we can relate the angular momentum of the solution (found from the metric near infinity) to the momentum of the solution, and check that these quantities satisfy the bound (1.4).

(d) Finally, we perform a sequence of dualities to map the winding and momentum charges of the solution to 5-brane and 1-brane charges respectively. This maps our solution to the one discussed recently in [6, 7], where the metric was found for the D1-D5 system with angular momentum. It was found in these references that the configuration with maximal angular momentum was in fact a smooth geometry with no singularities. We find that in a certain limit of parameters both the dual descriptions give geometries that are flat outside a ‘doughnut’ shaped tube of small thickness and large length. We discuss how such tubes may exhibit low energy collective oscillations, and the physical significance that such modes would have.

The plan of this paper is as follows. Section 2 investigates the profile for a classical string in flat space. In section 3 we derive the geometry of the spinning string by applying boosts and dualities to the neutral Kerr solution. Section 4 derives the same solution starting with the known metric of a single string, and superposing configurations to arrive at the multiwound string. Section 5 maps the spinning string solution to the D1-D5 system by dualities. Section 6 is a discussion.

2 Classical calculation for the rotating string.

Let us begin with a simple calculation which demonstrates that if we take a classical string (described by a Born-Infeld action) and excite vibrations on it that are purely right moving (thus giving a BPS state), then the ratio of angular momentum of the string to the momentum along the string has an upper bound. One of the directions, X^9 , of the 10-d flat spacetime is compactified on a circle:

$$X^9 \sim X^9 + 2\pi R. \tag{2.1}$$

The time direction is called X^0 . We assume that the string is wrapped n_w times around this circle before closing on itself. The momentum along the string will be called P . The string will in general have an angular momentum in the non-compact transverse directions $X^i, i = 1, \dots, 8$. This angular momentum is described by the tensor

$$M_{ij} = \frac{1}{2} \int d\sigma (X_i \mathcal{P}_j - X_j \mathcal{P}_i). \quad (2.2)$$

Here \mathcal{P}_i is the momentum density conjugate to X_i . One can characterize the value of angular momentum by the following invariant:

$$J = \sqrt{2M_{ij}M_{ij}}. \quad (2.3)$$

We will be interested only in a special case of BPS solutions which describe right-moving excitations of the string. We will show that for such solutions there is an upper bound on a ratio J/P :

$$\left| \frac{J}{P} \right| \leq R n_w. \quad (2.4)$$

The classical Dirac–Born–Infeld (Nambu–Goto) action for the string is:

$$S = -T \int d^2\sigma \sqrt{-\det g}, \quad (2.5)$$

where T is the tension of the string and g is the induced metric on the string worldsheet:

$$g_{ab} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} = \begin{pmatrix} \dot{X}^\mu \dot{X}_\mu & \dot{X}^\mu X'_\mu \\ \dot{X}^\mu X'_\mu & X'^\mu X'_\mu \end{pmatrix}. \quad (2.6)$$

Here X'_μ is the derivative of X_μ with respect to $\sigma^1 = \sigma$, while \dot{X}_μ is the derivative of X_μ with respect to $\sigma^0 = \tau$. In terms of X^μ the action (2.5) reads:

$$S = -T \int d^2\sigma \sqrt{(\dot{X}^\mu X'_\mu)^2 - (\dot{X}^\mu \dot{X}_\mu) (X'^\mu X'_\mu)}. \quad (2.7)$$

One can use this action to evaluate the canonical momenta:

$$P_\mu = \frac{\delta S}{\delta \dot{X}^\mu} = T \int \frac{d\sigma}{\sqrt{(\dot{X}^\mu X'_\mu)^2 - (\dot{X}^\mu \dot{X}_\mu) (X'^\mu X'_\mu)}} \left\{ \dot{X}_\mu (X'^{\nu'} X'_\nu) - X'_\mu (\dot{X}^\nu X'_\nu) \right\}. \quad (2.8)$$

We will need this expression later.

Let us now impose the static gauge. We want to have $0 \leq \sigma < 2\pi$. Since the string has winding number n_w we have

$$X^9(\sigma + 2\pi) = X^9(\sigma) + 2\pi R n_w. \quad (2.9)$$

Thus the static gauge condition is

$$X^0 = \tau, \quad X^9 = Rn_w \sigma. \quad (2.10)$$

In this gauge the action reads:

$$S = -T \int d^2\sigma \sqrt{R^2 n_w^2 - R^2 n_w^2 \dot{\mathbf{X}}^2 + \mathbf{X}'^2 - \dot{\mathbf{X}}^2 \mathbf{X}'^2 + (\dot{\mathbf{X}} \cdot \mathbf{X}')^2}. \quad (2.11)$$

Here \mathbf{X} is the eight dimensional vector (X^1, \dots, X^8) and the scalar product is defined in the usual way:

$$\mathbf{C} \cdot \mathbf{D} = \sum_{i=1}^8 C^i D^i \quad (2.12)$$

We are looking for solutions that describe purely right moving vibrations of the string. These vibrations will move at the speed of light, so \mathbf{X} has the form

$$\mathbf{X}(\tau, \sigma) = \mathbf{X}(\tilde{\sigma}), \quad \tilde{\sigma} = \sigma - \frac{\tau}{Rn_w}. \quad (2.13)$$

(It can be readily verified that such an \mathbf{X} satisfies the equations of motion.) In particular, this ansatz allows us to write the τ derivative in terms of the derivative with respect to σ :

$$\dot{\mathbf{X}} = -\frac{1}{Rn_w} \mathbf{X}' \quad (2.14)$$

For such excitations the square root in (2.11) simplifies to Rn_w , and we get for the momentum (2.8) along the direction X^9 :

$$P_9 = -T \int d\sigma \dot{\mathbf{X}} \cdot \mathbf{X}' = \frac{T}{Rn_w} \int d\sigma (\mathbf{X}')^2. \quad (2.15)$$

For the angular momentum (2.2) in the non-compact directions one finds:

$$M_{ij} = -\frac{T}{2} \int d\sigma (X_i X'_j - X_j X'_i). \quad (2.16)$$

Since $X_i(\tilde{\sigma} + 2\pi) = X_i(\tilde{\sigma})$, we can write

$$X_i(\tilde{\sigma}) = C_i(0) + \sum_{n=1}^{\infty} (C_i(n) \cos n\tilde{\sigma} + D_i(n) \sin n\tilde{\sigma}). \quad (2.17)$$

Substituting this expansion into (2.15), we find:

$$P_9 = \frac{T\pi}{Rn_w} \sum_{i=1}^8 \sum_{n=1}^{\infty} n^2 (C_i^2(n) + D_i^2(n)). \quad (2.18)$$

For the angular momentum M_{ij} one gets:

$$M_{ij} = -\frac{T\pi}{2} \sum_{n=1}^{\infty} 2n [C_i(n)D_j(n) - C_j(n)D_i(n)], \quad (2.19)$$

and for J (see (2.3)):

$$J = T\pi \left(\sum_{m,n=1}^{\infty} 4mn \{ \mathbf{C}(n) \cdot \mathbf{C}(m) \quad \mathbf{D}(n) \cdot \mathbf{D}(m) - \mathbf{C}(n) \cdot \mathbf{D}(m) \quad \mathbf{D}(n) \cdot \mathbf{C}(m) \} \right)^{1/2} \quad (2.20)$$

Let us now compare this expression with P_9 :

$$P_9 = \frac{T\pi}{Rn_w} \left(\sum_{m,n=1}^{\infty} m^2 n^2 (\mathbf{C}^2(n) + \mathbf{D}^2(n)) (\mathbf{C}^2(m) + \mathbf{D}^2(m)) \right)^{1/2} \quad (2.21)$$

Using the inequality

$$(\mathbf{a}^2 + \mathbf{c}^2)(\mathbf{b}^2 + \mathbf{d}^2) \geq 4 |(\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{c} \cdot \mathbf{b})|, \quad (2.22)$$

we find that:

$$|J| < |P_9 n_w R| \quad (2.23)$$

In the above relation we obtain equality of the two sides if and only if the coefficients $C_i(n), D_i(n)$ satisfy two conditions:

(a) Only the lowest harmonics of vibration should be excited; i.e. $\mathbf{C}(n) = \mathbf{D}(n) = 0$ for $n \neq 1$. ($C_i(0)$ can be set to zero by a translation.)

(b) The coefficients of these lowest harmonics should satisfy the relations

$$\mathbf{C}(1) \cdot \mathbf{D}(1) = 0, \quad (\mathbf{C}(1))^2 = (\mathbf{D}(1))^2. \quad (2.24)$$

These relations imply that in the inequality (2.22) we get $\mathbf{a} = \mathbf{b}, \mathbf{c} = \mathbf{d}$ and that $\mathbf{a} \cdot \mathbf{d} = \mathbf{c} \cdot \mathbf{b} = 0$.

Thus we get the following geometric picture of the string which carries the maximal possible J for a given P . The two vectors \mathbf{C}, \mathbf{D} give a 2-dimensional plane transverse to the direction X^9 ; without loss of generality we can let these be the directions X^1, X^2 . The string describes a helix, describing a circle in this $X^1 - X^2$ plane while moving up in the direction X^9 . The string closes back on itself only after n_w turns around the compact direction X^9 , so at any given value of X^9 we see n_w evenly spaced strands of the string. If n_w is large, then the string covers very densely the toroidal surface given by the circle $(X^1)^2 + (X^2)^2 = \text{constant}$ times the compact circle in X^9 .

Note that if instead of the lowest harmonic modes $C_i(1), D_i(1)$ we excite the m th harmonic modes $C_i(m), D_i(m)$ (while maintaining conditions analogous to (2.24)) then we will get

$$\left| \frac{J}{P} \right| = \frac{Rn_w}{m}, \quad (2.25)$$

We note that the above relations for J do not depend on the string tension T . The above conclusions hold unchanged for excitations of a p-brane that are independent of $p - 1$ of the spatial directions along the brane – this gives an ‘effective string’ along the remaining direction.

3 Generating the metric.

We saw above that for a classical string in flat space, if the winding number n_w is large, and the string carries a right moving wave with the largest possible angular momentum, then the strands of the string cover densely a torus given by a circle $(X^1)^2 + (X^2)^2 = a^2$ times the circle X^9 . We now want to write down the classical metric and gauge field produced by such a string configuration. In the limit of large n_w the mass density of the string is essentially uniform over this torus. There are many ways to extract the corresponding metric from known results in the literature², but it is easiest for our purposes to derive this metric from first principles by the following method³.

We start with a black hole in 4+1 dimensions x^0, x^1, \dots, x^4 carrying mass and angular momentum, but no charge. (We will use lower case letters for the curvilinear coordinates involved in metric computations, in contrast to the upper case letters used for the computations in flat space. In particular note that in the metric computations the direction of the string is x^5 while in the flat space computation it was X^9 .) We can regard this solution as a black brane solution in 10 spacetime dimensions, which is translationally invariant in the extra 5 directions x^5, \dots, x^9 . We boost this solution in the direction x^5 , thus getting a solution with mass, angular momentum and momentum charge. We then do a T-duality along x^5 , so that the momentum charge becomes an elementary string winding charge. We can now boost again in the x^5 direction, getting a solution that now has mass, angular momentum, as well as winding and momentum charges. Now taking the initial mass of the hole to zero gives an extremal BPS solution. This solution describes an elementary string that has winding and momentum along x^5 , as well as angular momentum in the transverse noncompact directions x^1, \dots, x^4 .

Note that we have let the solution be translationally invariant in the four directions $x^6 \dots x^9$. This invariance corresponds to taking a smearing the strings in the directions $x^6 \dots x^9$. Such a smearing makes no essential change to the computation (the powers of r in the various metric coefficients will reflect the number of noncompact directions, but the qualitative physics is unchanged). With this translational invariance we will be able to map the solution by dualities to a solution of D5 and D1 branes which is relevant to the 4+1 dimensional black hole problem and which has been studied recently in [6, 7].

Notation: We will be using the following notation to keep track of dualities and their effect on branes. T duality along the direction x^k will be denoted T_k , while S duality

²In particular we may obtain the metric as a special case of the result in [8][9].

³Such method was used for example to generate solutions for branes and Schwarzschild black holes in [10]

will be just called S . If we apply the duality T_k followed by S then the operation will be written as $T_k S$. A fundamental string extending in the directions t, x^k will be denoted $F1(k)$, and a Dp brane extending along $t, x^{i_1}, \dots, x^{i_p}$ will be called $Dp(i_1, \dots, i_p)$. The Neveu–Schwarz 5–brane spanning directions $t, x^5, x^6, x^7, x^8, x^9$ will be denoted as $NS5(5, 6, 7, 8, 9)$. Momentum charge along the direction x^k will be denoted $P(k)$.

3.1 Generating the metric for the rotating string.

As our starting point we take the metric for the five dimensional rotating black hole. In five dimensions the rotation is specified by two parameters, but we set one of them to be zero. Thus we start from the following metric (see for example [8, 11]):

$$ds_5^2 = -\frac{r^2 + a^2 \cos^2 \theta - 2m}{r^2 + a^2 \cos^2 \theta} dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2 - 2m} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 \\ + \left(r^2 + a^2 + \frac{2ma^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \right) \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\psi^2 - \frac{4ma \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} dt d\phi \quad (3.1)$$

Since this metric solves Einstein’s equations in the vacuum, one can consider it as a solution of low dimensional supergravity coming from either IIA or IIB string theory. To lift the solution to ten dimensions one can add five flat directions to the black hole (3.1). The resulting metric will satisfy ten dimensional Einstein equations in the vacuum, and thus it will also be a solution of type II supergravity with following values of NS–NS fields:

$$ds^2 = ds_5^2 + \sum_{i=5}^9 dx^i dx^i, \quad e^{2\Phi} = 1, \quad B_{\mu\nu} = 0, \quad (3.2)$$

while all RR fields vanish. In this paper we will always write metrics in the string frame unless specified otherwise. In particular, the metric in (3.2) can be thought of as a metric in the string frame (since the dilaton vanishes, there is no difference between the string and Einstein frames).

We will finally be interested in the case when directions x^5, \dots, x^9 are compactified on a torus:

$$x^5 \sim x^5 + 2\pi R_5, \quad \dots \quad x^9 \sim x^9 + 2\pi R_9. \quad (3.3)$$

The classical metric does not depend on whether or not we compactify a direction that is translationally invariant. Thus we can lift this solution to the covering noncompact space, apply a boost, and then decide to compactify the solution again along some other translationally invariant direction.

A boost in the direction x^5 is parameterized by one number z :

$$\begin{pmatrix} t' \\ x^{5'} \end{pmatrix} = \begin{pmatrix} \cosh z & \sinh z \\ \sinh z & \cosh z \end{pmatrix} \begin{pmatrix} t \\ x^5 \end{pmatrix} \quad (3.4)$$

To perform the boost we just write the metric (3.2) in terms of $t', x^{5'}$, and assume that the coordinate $x^{5'}$ is the new compact coordinate.

After this boost the solution is characterized by three parameters: m , a and z . Since it carries momentum charge along x^5 we denote it by P(5). Ultimately we will be interested in the limit of infinite boost, i.e. will take the mass parameter of the metric to zero while keeping the combination $A = \sqrt{m} \cosh z$ fixed. This can be achieved in two different ways: z can go to either positive or negative infinity. To account for these two possibilities we will characterize the boost (3.4) by two parameters:

$$A = \sqrt{m} \cosh z, \quad \alpha = \text{sign}(z). \quad (3.5)$$

To prevent a proliferation of superscripts, we relabel coordinates so that x'^5 is now called x^5 . (We have no need for the old coordinate x^5 .) Next we apply T-duality in the direction now called x^5 . The P(5) charge changes to the charge of fundamental string (F1(5) in our notation). The solution continues to have the mass parameter m and the rotation parameter a .

Let us now apply a new boost in the x^5 direction:

$$\begin{pmatrix} t' \\ x^{5'} \end{pmatrix} = \begin{pmatrix} \cosh w & \sinh w \\ \sinh w & \cosh w \end{pmatrix} \begin{pmatrix} t \\ x^5 \end{pmatrix}. \quad (3.6)$$

As before we can characterize this boost by the alternative parameters:

$$B = \sqrt{m} \cosh w, \quad \beta = \text{sign}(w). \quad (3.7)$$

Since the boost does not affect the F1 charge, it only produces a new momentum charge P(5), and the resulting solution can be labeled [F1(5)][P(5)].

Since coordinate x^5 plays a special role in our construction it is convenient to introduce a new notation for it:

$$y = x^5, \quad (3.8)$$

We now take the limit $m \rightarrow 0$, while keeping A and B fixed. It is helpful to introduce the functions

$$f_A = r^2 + a^2 \cos^2 \theta + 2A^2 \quad \text{and} \quad f_0 = r^2 + a^2 \cos^2 \theta. \quad (3.9)$$

(Note that f_0 is just f_A for $A = 0$.) Then the extremal rotating metric with charges [F1(5)][P(5)] obtained after these steps is

F1(5)P(5):

$$\begin{aligned} ds^2 &= -\frac{f_0}{f_A}(dt^2 - dy^2) + \frac{2B^2}{f_A}(dt - \beta dy)^2 - \frac{4aAB}{f_A} \sin^2 \theta d\phi(dt - \beta dy) \\ &+ f_0 \left(\frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + (a^2 + r^2) \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\psi^2 + \sum_{i=6}^9 dx^i dx^i \\ e^{2\Phi} &= \frac{f_0}{f_A}, \quad B_{t\phi} = -\beta B_{y\phi} = \frac{2aAB\alpha\beta \sin^2 \theta}{f_A}, \quad B_{ty} = -\frac{2A^2\alpha}{f_A} \end{aligned} \quad (3.10)$$

As mentioned above this metric and other metrics written below are string metrics unless specified otherwise.

4 Superposing solutions of a vibrating string

In the above section we derived the supergravity solution we wanted, but we cannot see from this solution itself the bound on the value of angular momentum that was derived for the classical string in section 2. The reason for can be seen from (2.25). We can see the value of J and P from the classical metric, but this ratio determines, at the microscopic level, not the integers n_w and m but only their ratio n_w/m . By taking n_w large, m large, we can make this ratio as close to any real number that we want, and thus we should find that in the classical solution even after J and P are fixed there is one more parameter in the solution. This extra one parameter freedom shows up in the parameter a .

Now we would like to see more geometrically the relation of the classical solution to the integer n_w . To do this we derive the classical supergravity solution found above in a different way. We consider the metric of a vibrating string, and superpose such strings to obtain a solution with the desired symmetries. In this process we will in fact start by choosing the values of n_w and m (we will take $m = 1$), and so obtain a direct map to the microscopic variables.

4.1 The metric of a vibrating string

The supergravity solution for a string carrying purely right moving vibrations is well known [12, 13]. We will follow for the most part notations of [12]. In line with the calculation of the above section, we smear the string over 4 directions $x_6 \dots x_9$; thus all quantities are independent of these coordinates. Then the solution is

$$\begin{aligned} ds^2 &= -e^{2\Phi} du dv - (e^{2\Phi} - 1) \dot{\vec{F}}^2 dv^2 + 2(e^{2\Phi} - 1) \dot{\vec{F}} \cdot d\vec{y} dv + d\vec{y} \cdot d\vec{y} + \sum_{i=6}^9 dx^i dx^i, \\ B_{uv} &= \frac{1}{2}(e^{2\Phi} - 1), \quad B_{vi} = \dot{F}_i(e^{2\Phi} - 1), \quad e^{-2\Phi} = 1 + \frac{Q}{|\vec{y} - \vec{F}|^2}. \end{aligned} \quad (4.1)$$

This solution is parameterized by the F1 charge Q and four dimensional vector $\vec{F} = (F_1, F_2, F_3, F_4)$, which is a function of only one coordinate v . \dot{F} denotes the derivative of F with respect to this coordinate v . As one can see, the dilaton $e^{2\Phi}$ goes to zero on the surface

$$\vec{y} = \vec{F}(v), \quad (4.2)$$

Thus this surface is interpreted as the location of the string. If $\vec{F} = \text{constant}$, the solution describes a static string, otherwise the string is oscillating.

4.2 The chiral null models

An interesting property of such string solutions is that we can easily construct the solution that describes a set of such strings instead of just one string. The wave on each string

must be carrying momentum in the same direction, but the waveforms need not be the same on different strings. To see how this happens, and to construct these multi-string solutions, it is helpful to review the chiral null models, of which the above string solution (4.1) is a special case.

Consider the following supergravity solution describing a chiral null model [14]:

$$\begin{aligned} ds^2 &= H(\vec{y}) \left(-du \, dv + K(\vec{y}, v) dv^2 + 2A_i(\vec{y}, v) dy_i dv \right) + d\vec{y} \cdot d\vec{y} + \sum_{i=6}^9 dx^i dx^i, \\ B_{uv} &= -G_{uv} = \frac{1}{2} H(\vec{y}), \quad B_{vi} = -G_{vi} = -H(\vec{y}) A_i(\vec{y}, v), \quad e^{-2\Phi} = H^{-1}(\vec{y}). \end{aligned} \quad (4.3)$$

Regarding A_i as a gauge field we can construct the field strength $\mathcal{F}_{ij} = A_{j,i} - A_{i,j}$. The functions in the chiral null model are required satisfy the equations

$$\partial^2 H^{-1} = 0, \quad \partial^2 K = 0, \quad \partial_i \mathcal{F}^{ij} = 0. \quad (4.4)$$

Here ∂^2 is the Laplacian in the y_i coordinates. Note that the indices i, j span the subspace $\{y_i\}$ where the metric is just δ_{ij} , and thus these indices are raised and lowered by this flat metric.

The above solution (4.1) for the oscillating string is a special case of such a chiral null model. To see this use the gauge freedom in B_{uv} to add a constant $1/2$ to the value B_{uv} in (4.1). Then we find that the string solution (4.1) gives a chiral null model with the following choice of functions

$$H^{-1}(\vec{y}, v) = 1 + \frac{Q}{|\vec{y} - \vec{F}|^2}, \quad K(\vec{y}, v) = \frac{Q|\dot{\vec{F}}|^2}{|\vec{y} - \vec{F}|^2}, \quad A_i(\vec{y}, v) = -\frac{Q\dot{F}_i}{|\vec{y} - \vec{F}|^2}. \quad (4.5)$$

4.3 The FP solution (3.10) as a chiral null model

In section 3 we had obtained the solution (3.10) with winding charge, momentum and angular momentum by a combination of boosts and dualities. In order to facilitate comparison with the rotating string solution, we write the solution (3.10) as a chiral null model also.

Introducing the null coordinates $u = t + y$, $v = t - y$, the metric of (3.10) reads:

$$\begin{aligned} ds_{FP}^2 &= -e^{2\Phi} du \, dv - \frac{B^2}{A^2} (e^{2\Phi} - 1) dv^2 + \frac{4aB}{A} (e^{2\Phi} - 1) \sin^2 \theta d\phi dv \\ &+ f_0 \left(\frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + (a^2 + r^2) \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\psi^2 + \sum_{i=6}^9 dx^i dx^i \end{aligned} \quad (4.6)$$

(Recall that $f_0 = r^2 + a^2 \cos^2 \theta$.) By a change of coordinates

$$r' = \sqrt{r^2 + a^2 \sin^2 \theta}, \quad \cos \theta' = \frac{r \cos \theta}{\sqrt{r^2 + a^2 \sin^2 \theta}}, \quad \phi' = \phi, \quad \psi' = \psi \quad (4.7)$$

the last line of the metric in (4.6) can be transformed into a flat metric:

$$\begin{aligned} f_0 \left(\frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + (a^2 + r^2) \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\psi^2 + \sum_{i=6}^9 dx^i dx^i \\ = dr'^2 + r'^2 d\theta'^2 + r'^2 \sin^2 \theta' d\phi'^2 + r'^2 \cos^2 \theta' d\psi'^2 + \sum_{i=6}^9 dx^i dx^i \end{aligned} \quad (4.8)$$

Introducing the Cartesian coordinates:

$$\begin{aligned} y_1 &= r' \sin \theta' \cos \phi', & y_2 &= r' \sin \theta' \sin \phi', \\ y_3 &= r' \cos \theta' \cos \psi', & y_4 &= r' \cos \theta' \sin \psi' \end{aligned} \quad (4.9)$$

and the following useful combinations:

$$z = \sqrt{y_1^2 + y_2^2}, \quad w = \sqrt{y_3^2 + y_4^2}. \quad (4.10)$$

the FP solution (4.6) reads:

$$\begin{aligned} ds^2 &= e^{2\Phi} \left(-du dv + \frac{B^2}{A^2} (e^{-2\Phi} - 1) dv^2 - \frac{4aB}{A} (e^{-2\Phi} - 1) \frac{y_1 dy_2 dv - y_2 dy_1 dv}{z^2 + w^2 + a^2 + f_0} \right) \\ &+ d\vec{y} d\vec{y} + \sum_{i=6}^9 dx^i dx^i, \\ B_{uv} &= -\frac{1}{2} (e^{2\Phi} - 1) \sim G_{uv}, \quad B_{vi} = -G_{vi}, \quad e^{-2\Phi} = 1 + \frac{2A^2}{f_0}. \end{aligned} \quad (4.11)$$

The function f_0 is given by the following expression:

$$f_0 = \left((z^2 - a^2)^2 + 2w^2(z^2 + a^2) + w^4 \right)^{1/2}. \quad (4.12)$$

The singularity of the solution is located on the surface where the dilaton $e^{2\Phi}$ vanishes, i.e. on the surface $f_0 = 0$. On the transverse space $y_1 \dots y_4$ this surface looks like a circle:

$$w = 0, \quad z = a. \quad (4.13)$$

Thus at any fixed value of time t the surface where the dilaton vanishes is a 6-dimensional spatial surface: four directions $x_6, \dots x_9$ over which the string is smeared, the coordinate y along the string and the direction along the circle $y_1^2 + y_2^2 = a^2$.

The solution (4.11) obtained above has a form of a chiral null model (4.3) with the following choice of functions:

$$\begin{aligned} H^{-1} &= 1 + \frac{2A^2}{f_0}, & K &= \frac{2B^2}{f_0}, \\ A_1 &= \frac{4ABay_2}{f_0(z^2 + w^2 + a^2 + f_0^2)}, & A_2 &= -\frac{4ABay_1}{f_0(z^2 + w^2 + a^2 + f_0^2)}, \quad A_3 = A_4 = 0 \end{aligned} \quad (4.14)$$

4.4 Superposing string solutions

Let us return to the supergravity solution (4.1) for an oscillating string, and its representation (4.5) as a chiral null model. We will now superpose such string solutions to obtain the same chiral null model (4.11) obtained above.

This superposition will proceed in two steps. First consider a single string wrapped in a periodic fashion around the compact coordinate y . Then in the functions H, K, A_i appearing in the chiral null model we will find that for any fixed value of v we have a ‘single center’ solution (4.5) of the Laplacian in the transverse space $\{y_i\}$. But we could let the string be ‘multiwound’ around the compact direction y , which means that it closes only after several turns around this direction. Such a string can be constructed by choosing the functions \vec{F} in (4.1) in the following manner

$$F_1(v) = a \cos(\omega v + \alpha), \quad F_2(v) = a \sin(\omega v + \alpha). \quad (4.15)$$

where

$$\omega = \frac{m}{n_w} \frac{1}{R} \quad (4.16)$$

If m/n_w is not an integer, then the string does not close on itself after one cycle around the direction y , but does close after a finite number of revolutions. For any given value of y , we will see several strands of the string, evenly spaced around the circle $y_1^2 + y_2^2 = a^2$. Such a configuration is a solution of the chiral null model where the harmonic function H^{-1} is a ‘multi-centered’ solution of the Laplace equation.

Though at this stage we have several centers of the string on the circle $y_1^2 + y_2^2 = a^2$ we do not for any finite n_w have translational invariance along this circle. To obtain a translationally invariant solution we ‘smear’ the multiwound string (characterized by the integers m, n_w) on this circle. One can think of this smearing operation as one where we cut up the multiwound string into thinner strands, each wound around y in the same way as the original strand, but with the parameter α in (4.15) taking a different value for each of these thin strands. In the limit where we take a uniform smearing, we just average the angle α with uniform weight over the circle $y_1^2 + y_2^2 = a^2$. Note that because of the linearity of the chiral null model we still get an exact solution of the supergravity equations, and that this solution reflects the value of the integers m, n_w through the value of ω which is a parameter that will appear in the smeared solution. The smearing operation gives

$$\begin{aligned} \langle H^{-1}(\vec{y}) \rangle &= 1 + \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\alpha}{(y_1 - a \cos \alpha)^2 + (y_2 - a \sin \alpha)^2 + w^2} \\ &= 1 + \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\alpha}{z^2 + w^2 + a^2 - 2za \cos(\alpha + \omega v - \phi')} \\ &= 1 + \frac{Q}{f_0}, \end{aligned} \quad (4.17)$$

and similarly

$$\begin{aligned}
\langle K(\vec{y}, v) \rangle &= \frac{Qa^2\omega^2}{f_0}, \\
\langle A_1 \rangle &= \frac{Qa\omega}{2\pi} \int_0^{2\pi} \frac{d\alpha \sin(\alpha + \phi')}{z^2 + w^2 + a^2 - 2za \cos \alpha} = -\frac{2Qa^2\omega y_2}{f_0} \frac{1}{z^2 + a^2 + w^2 + f_0}, \\
\langle A_2 \rangle &= \frac{2Qa^2\omega y_1}{f_0} \frac{1}{z^2 + a^2 + w^2 + f_0}
\end{aligned} \tag{4.18}$$

Here we evaluated the integrals using the following result:

$$\int_0^{2\pi} \frac{d\alpha \cos^n \alpha}{1 + a \cos \alpha} = \frac{2\pi}{\sqrt{1 - a^2}} \left(\frac{\sqrt{1 - a^2} - 1}{a} \right)^n. \tag{4.19}$$

4.5 Comparing the solutions (4.14) and (4.17)–(4.18)

We have obtained the metric of a fundamental string carrying momentum and angular momentum in terms of a chiral null model, in two different ways. The solution (4.14) was obtained for the metric obtained by boosting and dualities starting with a Kerr black hole. This solution is characterized by the parameters a, A, B . The solution (4.17)–(4.18) was obtained by smearing the solution of a spinning string. This latter solution is characterized by parameters a, ω, Q . Comparing the functions in these chiral null model descriptions we find that the parameter a is the same in the two solutions, while the parameters ω, Q are given by

$$\begin{aligned}
Q &= 2A^2 \\
\omega &= -\frac{B}{aA}
\end{aligned} \tag{4.20}$$

4.6 Momentum and angular momentum of the solutions

Let us extract the momentum and angular momentum of the supergravity solutions that we have found, by looking at the behavior of the fields at infinity. We will first find these quantities at the classical level, and then proceed to find the integer number of number of quanta that correspond to each quantity.

4.6.1 Classical charges of the solution

The string metric of the 10-D FP solution (3.10) behaves at infinity as

$$\begin{aligned}
ds^2 &\approx -\left(1 - \frac{2A^2}{r^2}\right)(dt^2 - dy^2) + \frac{2B^2}{r^2}(dt - dy)^2 - \frac{4aAB}{r^2} \sin^2 \theta d\phi(dt - dy) \\
&+ dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2) + \sum_{i=6}^9 dx^i dx^i
\end{aligned} \tag{4.21}$$

$$e^{2\Phi} \approx 1 - \frac{2A^2}{r^2}, \quad B_{t\phi} = -B_{y\phi} \approx \frac{2aAB \sin^2 \theta}{r^2}, \quad B_{ty} \approx -\frac{2A^2}{r^2} \tag{4.22}$$

We should really extract the conserved quantities from the Einstein metric, but since we will be using parts of the metric that are vanish at infinity the correction we get from the presence of the dilaton is subleading, and does not affect the value of the conserved quantity. (By contrast, if we were computing the mass then we will have to use the Einstein metric explicitly.)

The momentum is extracted from the metric coefficient g_{ty} . Let a classical solution be translation invariant in the directions x_\perp , and let D be the number of remaining spacetime dimensions. The direction y is one of the directions in x_\perp . Then if r is the radial coordinate in the D dimensional spacetime, and if near infinity

$$g_{ty} \sim \frac{q}{r^{D-3}} \quad (4.23)$$

then the momentum of the solution in the direction y , per unit volume in the space x_\perp , is

$$\mathcal{P}_y = -\frac{(D-3)\Omega_{D-2}q}{16\pi G} \quad (4.24)$$

where Ω_{D-2} is the volume of the unit $D-2$ sphere and G is the Newton's constant of the entire spacetime (the D dimensional part together with the directions in x_\perp). Using this relation we get for our solution

$$\mathcal{P}_y = \frac{\pi B^2}{2G} \quad (4.25)$$

and the total momentum is

$$P_y = \frac{\pi B^2}{2G}(2\pi R)(2\pi)^4 V \quad (4.26)$$

where $2\pi R$ is the length of the direction y and $(2\pi)^4 V$ is the volume in the directions $x_6, \dots x_9$.

The angular momentum is extracted from the coefficient $g_{t\phi}$. If near infinity

$$g_{t\phi} \sim \frac{\tilde{q}}{r^{D-3}} \quad (4.27)$$

then the angular momentum in the direction ϕ per unit volume in x_\perp is

$$\mathcal{J}_\phi = \frac{\Omega_{D-2}\tilde{q}}{8\pi G} \quad (4.28)$$

We then find that for our solution

$$\mathcal{J}_\phi = -\frac{aAB\pi}{2G} \quad (4.29)$$

and the total angular momentum is

$$J_\phi = -\frac{aAB\pi}{2G}(2\pi R)(2\pi)^4 V \quad (4.30)$$

In terms of the parameters used in the description (4.17)–(4.18) we find (using (4.20))

$$\begin{aligned} P_y &= \frac{\pi a^2 Q \omega^2}{4G} (2\pi R) (2\pi)^4 V \\ J_\phi &= \frac{\pi a^2 Q \omega}{4G} (2\pi R) (2\pi)^4 V \end{aligned} \quad (4.31)$$

Thus

$$\frac{J_\phi}{P_y} = \frac{1}{\omega} \quad (4.32)$$

Using the value (4.16) of ω from the geometric picture of the string, we get

$$\frac{J_\phi}{P_y} = \frac{n_w R}{m} \quad (4.33)$$

in agreement with the classical result (2.25) from the flat space computation. This agreement is of course not a surprise – we just obtain a check on our gravity calculations since we have now computed the momentum and angular momentum from the metric produced by the string rather than from the string profile itself.

4.6.2 Charges in quantized units

In the 10-d string theory the Newton's constant is

$$G = g^2 8\pi^6 \alpha'^4 \quad (4.34)$$

where g is the string coupling. Let us set $\alpha' = 1$. Then quantum mechanically (4.26) gives

$$P = \frac{2B^2 R V}{g^2} = \frac{n_p}{R} \quad (4.35)$$

Substituting this in (4.33) gives

$$J_\phi = \frac{n_p n_w}{m} \quad (4.36)$$

as expected.

We can also find the number of strings from the gauge field $B_{\mu\nu}$. To do this one needs to evaluate the following flux over the three sphere (θ, ϕ, ψ) [15]:

$$n_w = \frac{V}{4\pi^2 g^2} \int_{S^3} e^{-\Phi} * H, \quad (4.37)$$

where Hodge dual is taken in 6 dimensions: $t, r, \theta, \phi, \psi, y$. The relevant pieces of H and its dual are:

$$H_{rty} \approx \frac{4A^2}{r^3}, \quad *H_{\theta\phi\psi} \approx \frac{r^6 \sin^2 \theta \cos^2 \theta}{\sqrt{r^6 \sin^2 \theta \cos^2 \theta}} H_{rty} = 4A^2 \sin \theta \cos \theta. \quad (4.38)$$

Thus the F1 charge n_w is given by:

$$n_w = \frac{4A^2 V}{4\pi^2 g^2} \int d\theta d\phi d\psi |\sin \theta \cos \theta| = \frac{2A^2 V}{g^2}, \quad (4.39)$$

5 Relation to F1–NS5 solution.

In the papers [6, 7] an interesting observation was made. Consider the D1-D5 system, where the D5 branes are wrapped on the torus composed of the compact directions x_5, x_6, \dots, x_9 , and the D1 branes are wrapped around x_5 . There are two ways to have angular momentum in this system. One way is to add momentum excitations along x_5 , with each momentum mode carrying a polarization in the noncompact directions $x_1 \dots x_4$, and thus contributing to the angular momentum. This kind of angular momentum was given to the system in the study of the 3-charge black hole [16, 17]. But the D1-D5 system can also carry angular momentum without any momentum excitations – the D1-D5 bound state itself can have a spin that ranges from zero to $n_1 n_5$, where n_1 and n_5 are the numbers of D1 branes and D5 branes respectively. One way to see this fact is to perform a set of S and T dualities (they will be listed below) to map the D1-D5 system to a set of elementary strings (wound around x_5) carrying momentum (along x_5). The latter system can, as was discussed in the preceding sections, carry angular momentum in the noncompact directions x_1, \dots, x_4 ; the magnitude of this angular momentum can range from zero to $n_1 n_5$.

The interesting observation of [6, 7] was that when the D1-D5 system (without momentum excitations) is placed in a configuration of maximal angular momentum $n_1 n_5$, then the corresponding classical geometry is completely smooth, with no horizon and no naked singularity. If the angular momentum is $\gamma n_1 n_5$, $\gamma < 1$, then there is still no horizon but there is a conical defect in the geometry at $r = 0$. Since the configuration with maximal angular momentum is a unique one, while those with lower angular momenta have some degeneracy, one may speculate that the interpretation of the conical singularity is that this singularity reflects the nontrivial entropy of states for $\gamma < 1$, and the absence of any singularity for $\gamma = 1$ reflects the uniqueness of that state.

Since the D1-D5 system can be mapped by dualities to the fundamental string carrying momentum, each configuration of the D1-D5 system (possessing some value of γ) can be mapped to some configuration of the spinning string of the kind that we have studied in the sections above. This leads to the natural question: which configurations of the spinning string are the ones dual to the smooth D1-D5 geometry with $\gamma = 1$? Can we see any analogue of this smoothness of the latter system in the corresponding string solution?

At first the analysis of this issue appears to pose a puzzle. The classical spinning string geometries are described by the following parameters. The string describes a helix which has a radius a and a height R ($a^2 = y_1^2 + y_2^2$, and R is the radius of the y coordinate around which the string is wrapped). In addition we have the parameters A, B which also have units of length and which describe the curvature length scales of the metric due to the string and the momentum respectively. (The solution also has the flat torus x_6, \dots, x_9 of volume $(2\pi)^4 V$, but this is not very relevant to the physics below.)

Thus there are three dimensionless ratios that can be constructed out of these four quantities R, a, A, B . At first one might think that some combination of these dimensionless ratios must map to the dimensionless parameter γ which describes the dual D1-D5

system. But if such is the case, then there must be a preferred subset of the spinning string geometries, which map to $\gamma = 1$. This appears puzzling, since it is not clear why any of the spinning string geometries should be more special than others.

The actual situation with the duality map is a little more complex. As we will see below, the value of γ in terms of the parameters of the spinning string is

$$\gamma = \frac{ag^2\alpha'^3}{2RABV} \quad (5.1)$$

where we have temporarily restored the string length squared parameter α' . (In all other relations α' is set to unity.) Let us set V to be order α'^2 for concreteness. Then we observe that if the parameters a, R, A, B are all macroscopic lengths, i.e. many times the string length $\sqrt{\alpha'}$, then γ is near *zero* if a, R, A, B are all lengths of comparable order. In other words if we hold the relative values of a, R, A, B fixed and take the classical limit, then any such system maps to a dual D1-D5 system with $\gamma = 0$. If, on the other hand we want $\gamma = 1$ then we will need to have the radius of the helix, a , much larger than the scales R, A, B . Thus the value $\gamma = 1$ is obtained in the spinning string picture as a configuration whose length ratios are not classical in the sense that they diverge as the Planck constant is taken to zero.

We will discuss the physics of such configurations further in the discussion. In the remainder of this section we construct explicitly the duality map between the F1-P system (the elementary string with momentum) and the NS5–F1 system (the elementary string and NS5–brane system, which is S-dual to the D1-D5 system); the metric for the latter is given for example in [8, 18, 6, 7].

First we introduce a convenient notation. Let us label the system of two kinds of charges as

$$\left| \begin{array}{c} P(5) \\ F1(5) \end{array} \right| \quad (5.2)$$

Application of S duality transforms $F1(5)$ into $D1(5)$ and leaves $P(5)$ invariant. We will use following diagram to describe the action of such a duality:

$$\left| \begin{array}{c} P(5) \\ F1(5) \end{array} \right| \xrightarrow{S} \left| \begin{array}{c} P(5) \\ D1(5) \end{array} \right| \quad (5.3)$$

The $P(5)$ – $F1(5)$ system is connected with $F1$ –NS5 by the following chain of dualities:

$$\left| \begin{array}{c} P(5) \\ F1(5) \end{array} \right| \xrightarrow{S} \left| \begin{array}{c} P(5) \\ D1(5) \end{array} \right| \xrightarrow{T6789} \left| \begin{array}{c} P(5) \\ D5(56789) \end{array} \right| \xrightarrow{S} \left| \begin{array}{c} P(5) \\ NS5(56789) \end{array} \right| \xrightarrow{T56} \left| \begin{array}{c} F1(5) \\ NS5(56789) \end{array} \right| \quad (5.4)$$

It is also important to see what happens with various parameters of the solution (e.g. coupling constant and radii) under such dualities. In previous sections we have introduced

the string coupling g , the radius R along x_5 , and the volume $(2\pi)^4 V$ of the torus T^4 in the directions $x_6 \dots x_9$. Let us define explicitly the radius in the direction x_6 through $(x^6 \sim x^6 + 2\pi R_6)$; we will need this radius in order to do T-dualities in x_6 . Note that in our notation the charges A and B are not invariant under S duality, but transform in the same way as coordinates (see appendix for details). We find the following transformations for the parameters:

$$\begin{pmatrix} g \\ A \\ R \\ R_6 \\ V \end{pmatrix} \xrightarrow{S} \begin{pmatrix} 1/g \\ A/\sqrt{g} \\ R/\sqrt{g} \\ R_6/\sqrt{g} \\ V/g^2 \end{pmatrix} \xrightarrow{T6789} \begin{pmatrix} g/V \\ A/\sqrt{g} \\ R/\sqrt{g} \\ \sqrt{g}/R_6 \\ g^2/V \end{pmatrix} \xrightarrow{S} \begin{pmatrix} V/g \\ A\sqrt{V}/g \\ R\sqrt{V}/g \\ \sqrt{V}/R_6 \\ V \end{pmatrix} \xrightarrow{T56} \begin{pmatrix} R_6/R \\ A\sqrt{V}/g \\ g/(R\sqrt{V}) \\ R_6/\sqrt{V} \\ R_6^2 \end{pmatrix} \quad (5.5)$$

Transformation laws for B and a are the same as for A .

After performing these dualities on the FP solution, one gets the following NS5-F1 (i.e. NS5-brane and fundamental string) solution (see appendix for the details):

$$\begin{aligned} ds^2 &= -\frac{\tilde{f}_0}{\tilde{f}_B}(dt^2 - d\tilde{y}^2) + \tilde{f}_A\left(\frac{dr^2}{r^2 + \tilde{a}^2} + d\theta^2\right) + (r^2 + 2\tilde{A}^2 - \frac{2\tilde{A}^2\tilde{a}^2 \cos^2 \theta}{\tilde{f}_B}) \cos^2 \theta d\psi^2 \\ &+ (r^2 + \tilde{a}^2 + 2\tilde{A}^2 + \frac{2\tilde{A}^2\tilde{a}^2 \sin^2 \theta}{\tilde{f}_B}) \sin^2 \theta d\phi^2 - \frac{4\tilde{A}\tilde{B}\tilde{a}}{\tilde{f}_B} \sin^2 \theta dt d\phi + \sum_{i=6}^9 dx^i dx^i \\ &- \frac{4\tilde{A}\tilde{B}\tilde{a}\alpha\beta}{\tilde{f}_B} \cos^2 \theta d\tilde{y} d\psi \end{aligned} \quad (5.6)$$

$$\begin{aligned} e^{2\Phi} &= \frac{\tilde{f}_A}{\tilde{f}_B}, \quad B_{ty} = -\frac{2\beta\tilde{B}^2}{\tilde{f}_B}, \quad B_{t\psi} = -\frac{2\tilde{A}\tilde{B}\tilde{a}\alpha \cos^2 \theta}{\tilde{f}_B}, \\ B_{y\phi} &= -\frac{2\tilde{A}\tilde{B}\tilde{a}\beta \sin^2 \theta}{\tilde{f}_B}, \quad B_{\phi\psi} = 2\tilde{A}^2\alpha \cos^2 \theta + \frac{2\tilde{A}^2\alpha\tilde{a}^2 \sin^2 \theta \cos^2 \theta}{\tilde{f}_B}. \end{aligned} \quad (5.7)$$

$$\tilde{f}_A = r^2 + \tilde{a}^2 \cos^2 \theta + 2\tilde{A}^2. \quad (5.8)$$

The coordinate x^5 has now been called \tilde{y} to remind ourselves that the compactification radii for the original coordinate y (in the FP solution) and \tilde{y} are different:

$$y \sim y + 2\pi R, \quad \tilde{y} \sim \tilde{y} + 2\pi \tilde{R}, \quad \tilde{R} = \frac{g}{R\sqrt{V}}. \quad (5.9)$$

The integer F1 and NS5 charges associated with this solution are given by,

$$\begin{aligned} n'_1 &= \frac{\tilde{V}}{4\pi^2 \tilde{g}^2} \int_{S^3} e^{-\Phi} * H = \frac{2\tilde{B}^2 \beta V'}{\tilde{g}^2} = \frac{2B^2 \beta R^2 V}{g^2} = n_P, \\ n'_5 &= \frac{1}{4\pi^2} \int_{S^3} H = 2\tilde{A}^2 \alpha = \frac{2A^2 \alpha V}{g^2} = n_w, \end{aligned} \quad (5.10)$$

while the angular momentum is the same as for the FP solution.

To compare the above NS5-F1 solution with the solution in [6, 7], we reduce the system (5.6) to six dimensions $(t, r, \theta, \phi, \psi, \tilde{y})$ and look at the resulting metric in the Einstein frame:

$$\begin{aligned}
ds_E^2 &= e^{-\frac{4\Phi}{6-2}} ds_6^2 = -\frac{\tilde{f}_0}{\sqrt{\tilde{f}_A \tilde{f}_B}} (dt^2 - d\tilde{y}^2) - \frac{4\tilde{A}\tilde{B}\tilde{a}}{\sqrt{\tilde{f}_A \tilde{f}_B}} (\sin^2 \theta dt d\phi + \alpha\beta \cos^2 \theta d\tilde{y} d\psi) \\
&+ \sqrt{\tilde{f}_A \tilde{f}_B} \left(\frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + \frac{\cos^2 \theta d\psi^2}{\sqrt{\tilde{f}_A \tilde{f}_B}} (\tilde{f}_A \tilde{f}_B - \tilde{a}^2 \cos^2 \theta (\tilde{f}_A + \tilde{f}_B - \tilde{f}_0)) \\
&+ \frac{\sin^2 \theta d\phi^2}{\sqrt{\tilde{f}_A \tilde{f}_B}} (\tilde{f}_A \tilde{f}_B + \tilde{a}^2 \sin^2 \theta (\tilde{f}_A + \tilde{f}_B - \tilde{f}_0))
\end{aligned} \tag{5.11}$$

Let us now set $\alpha = \beta = 1$, and compare the above solution with the solution presented in [6, 7]. We use the notation of [7]⁴:

$$\begin{aligned}
d\bar{s}^2 &= -\frac{\sqrt{Q_1 Q_5}}{h} (dt^2 - d\bar{y}^2) + \sqrt{Q_1 Q_5} h f \left(d\theta^2 + \frac{d\bar{r}^2}{\bar{r}^2 + \gamma^2} \right) \\
&- \frac{2\gamma \sqrt{Q_1 Q_5}}{h f} (\cos^2 \theta d\bar{y} d\psi + \sin^2 \theta dt d\phi) \\
&+ h \sqrt{Q_1 Q_5} \left[\left(\bar{r}^2 + \frac{\gamma^2 \cos^2 \theta}{h^2 f^2} \right) \cos^2 \theta d\psi^2 + \left(\bar{r}^2 + \gamma^2 - \frac{\gamma^2 \sin^2 \theta}{h^2 f^2} \right) \sin^2 \theta d\phi^2 \right],
\end{aligned} \tag{5.12}$$

$$f = \bar{r}^2 + \gamma^2 \cos^2 \theta, \quad h = \frac{1}{\bar{R}^2 f} (Q_5 f + \bar{R}^2)^{1/2} (Q_1 f + \bar{R}^2)^{1/2} \tag{5.13}$$

The metrics (5.11) and (5.12) are related by the change of variables

$$\bar{R} = \tilde{R}, \quad Q_1 = 2\tilde{B}^2, \quad Q_5 = 2\tilde{A}^2, \quad \gamma = \frac{\tilde{a}\tilde{R}}{2\tilde{A}\tilde{B}} \tag{5.14}$$

$$\bar{r} = \frac{r\tilde{R}}{2\tilde{A}\tilde{B}}, \quad \bar{t} = \frac{t}{\tilde{R}}, \quad \bar{y} = \frac{\tilde{y}}{\tilde{R}}. \tag{5.15}$$

One can also rewrite the expressions for the parameters of the solution (5.12) in terms of parameters of the FP solution (3.10) which was obtained for the spinning string:

$$\bar{R} = \frac{g}{R\sqrt{V}}, \quad Q_1 = \frac{2B^2 V}{g^2}, \quad Q_5 = \frac{2A^2 V}{g^2}, \quad \gamma = \frac{ag^2}{2RABV} = \frac{J}{n_1 n_P} \tag{5.16}$$

⁴The solution of [7] is for the D5-D1 system rather than NS5-F1, but since we are looking only at the Einstein metric there is no change under the S-duality that relates these two systems.

6 Discussion

We have found the supergravity solution which describes a string that is wound several times around a compact direction and carries a right moving wave of the form (4.15). This solution was obtained by applying boosts and T-dualities to the neutral Kerr solution, and also by superposing the known solution of the oscillating string. The momentum and angular momentum of the solution were read off from the metric, and related to the corresponding values of these quantities expected from a classical computation in flat space. The metric of the string with momentum is dual to a D1-D5 system; this duality map was constructed explicitly.

While there is nothing very novel about finding such supergravity solutions, there are several reasons why we would like to study the properties of this solution in some detail. If we study absorption by a D1-D5 system at weak coupling, the energy of an incoming quantum gets converted to the energy of vibrations that run up and down the ‘effective string’ formed by the 1-brane - 5-brane system. If we increase the coupling to obtain a black hole geometry, then such a model may not hold – the left and right moving modes may not remain weakly interacting. One can then try to describe the system by a dual $AdS_3 \times S^3 \times M_4$ geometry [19], but in the latter picture we do not, so far, have an idea of how an incoming quantum actually returns back to asymptotic infinity as (information carrying) Hawking radiation.

The consideration of angular momentum may provide a domain where some aspects of both descriptions survive. Consider the string with momentum charge and large angular momentum. Let us take the size R of the circle on which the string is wrapped to be of order the string length, and take the compact torus $x_6 \dots x_9$ to be of string size as well. If the angular momentum is of order the maximal allowed value $J_{\max} = n_w n_p$, then using (5.1) we see that for any given g , as n_p, n_w become large, the length scale a becomes much larger than the length scales A, B (which describe the curvature length scales of the metric due to the winding and momentum charges respectively). From the geometry (3.10) we see that the torus on which the string is wrapped becomes very small in one direction (the direction of length R) and very long in its other direction (the circle of radius a). The string location is characterized by the region where the dilaton becomes small, and this can be seen to be $r \approx 0$, $\theta \approx \pi/2$. Away from this region, the first line of the metric in (3.10) is just $-(dt^2 - dy^2)$, while the second line is always a flat space metric as shown by the relations (4.7)–(4.8). Thus we see that we obtain a thin and long doughnut shaped ‘tube’, sitting in a spacetime that is otherwise approximately flat. The thickness of the tube can be characterized by $\max\{A, B\}$, which describes how far the gravitational field extends from the axis of the tube. The circumference of this tube is long: $2\pi a$. In the coordinates (4.7) this tube is located at

$$r' \approx a, \quad \theta \approx \pi/2, \quad 0 \leq \phi < 2\pi. \quad (6.1)$$

What will be the description of low energy excitations of this multiply wrapped string? Imagine a vibration of the string in the compact direction x_6 . The wavelength of the

vibration is order $2\pi n_w R$, so the deformation δx_6 does not change significantly as we go once around the circle in the direction y : $y \rightarrow y + 2\pi R$. But it does change as we go halfway around the string, i.e. increase ϕ by an amount of order unity. So δx_6 is essentially only a function of ϕ and not a function of y . Thus we see that long wavelength vibrations of the string look like low frequency vibrations of the effective ‘tube’ (6.1). We may thus write an effective action for this tube, and investigate its oscillations. We will carry out such an investigation elsewhere, but here we note that if this tube has long lived oscillations that are superpositions of left and right moving modes, then we would have a situation where at strong coupling (i.e. in a supergravity solution) we have obtained the non-BPS state describing both left and right vibrations of a string. For this string then we would have a common description of (non-BPS) excitations both in the weak coupling (flat space) and strong coupling (supergravity) domains.

Let us turn now the smooth metric found in [6, 7] to describe the D5-D1 system at maximum angular momentum ($\gamma = 1$). This metric has three essential length scales: the curvature scales \tilde{A}, \tilde{B} which arise from the D5 and D1 charges respectively, and the length \tilde{R} which is the length of the \tilde{y} circle at infinity. We set $\tilde{A} = \tilde{B}$, and look at the ratio \tilde{R}/\tilde{A} . When $\tilde{R}/\tilde{A} \gg 1$ the geometry has a large region of the form $AdS_3 \times S^3$; some energy scales for this geometry were investigated in [20]. Here we wish to consider the opposite limit: $\tilde{R}/\tilde{A} \ll 1$. In this case it can be checked that the geometry of [6, 7], given in (5.12), is essentially flat space outside a ‘tube’ that has the shape of a thin doughnut. The thickness of the tube is $\sim \tilde{A}$ and the length of the tube is $2\pi a$. Mapping this geometry by the dualities (5.5) to the oscillating string FP geometry, we find again that the string is wrapped on the surface of a similar ‘doughnut’, and spacetime is essentially flat outside the doughnut. The only difference between the two pictures is that in the D5-D1 geometry the spacetime inside the doughnut is smooth, while in the FP solution the microscopic description of the doughnut is in terms of closely spaced strands of an elementary string. (Note however that the smoothness of the spacetime in the former case is only a result at the level of classical supergravity: if there was a residual conical defect of order $1/n_5$ as $\gamma \rightarrow 1$ then the classical limit would still indicate a smooth spacetime at $\gamma = 1$.) It would be helpful to investigate further the low energy excitations of the D1-D5 system in this limit by looking at correlators at the orbifold point of the D1-D5 CFT [21].

A Dualities in string theory: conventions and notations.

In section 3 we started from fundamental string with some momentum along it and applied a chain of string dualities in order to construct the NS5–F1 system. In this appendix we summarize the rules describing the action of various dualities on matter fields. We use the same conventions as [22].

We begin with the actions for type II supergravities, which fix normalization of the fields. The bosonic part of the action for type IIA theory is:

$$S_{IIA} = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-G} \left\{ e^{-2\Phi} \left[R + 4(\nabla\phi)^2 - \frac{1}{12} (H^{(3)})^2 \right] - \frac{1}{4} (G^{(2)})^2 - \frac{1}{48} (G^{(4)})^2 \right\} - \frac{1}{2(2\pi)^7 \alpha'^4} \int B^{(2)} dC^{(3)} dC^{(3)}. \quad (A.1)$$

The field strengths of NS field B and RR fields C are given by:

$$H^{(3)} = dB^{(2)}, \quad G^{(2)} = dC^{(1)}, \quad G^{(4)} = dC^{(3)} + H^{(3)} \wedge C^{(1)}. \quad (A.2)$$

For the type IIB theory we have:

$$S_{IIB} = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-G} \left\{ e^{-2\Phi} \left[R + 4(\nabla\phi)^2 - \frac{1}{12} (H^{(3)})^2 \right] - \frac{1}{12} (G^{(3)} + C^{(0)} H^{(3)})^2 - \frac{1}{2} (dC^{(0)})^2 - \frac{1}{480} (G^{(5)})^2 \right\} + \frac{1}{2(2\pi)^7 \alpha'^4} \int \left(C^{(4)} + \frac{1}{2} B^{(2)} C^{(2)} \right) G^{(3)} H^{(3)}. \quad (A.3)$$

In this case the RR field strengths are defined by:

$$G^{(3)} = dC^{(2)}, \quad G^{(5)} = dC^{(4)} + H^{(3)} \wedge C^{(2)}. \quad (A.4)$$

Here G is the string metric. The Einstein metric is defined through

$$G_{\mu\nu}^E = e^{(\Phi_0 - \Phi)/2} G_{\mu\nu} = \sqrt{g} e^{-\Phi/2} G_{\mu\nu}. \quad (A.5)$$

(This gives the 10-d Newton's constant as $16\pi G_N = (2\pi)^7 \alpha'^4 g^2$.) Here Φ_0 is the asymptotic value of the dilaton. We will also need a generalization of (A.5) for toroidal compactifications. We assume that the ten dimensional space is a product $M^D \times T^{10-D}$, and all fields are independent of the coordinates on the torus [23]. The D-dimensional dilaton is given by:

$$\phi = \Phi - \frac{1}{2} \log \bar{G}, \quad (A.6)$$

where \bar{G} is the part of the metric G describing internal space T^{10-D} . Then the D dimensional metric in the Einstein frame is given by:

$$G_{\mu\nu}^E = e^{-4(\phi - \phi_0)/(D-2)} G_{\mu\nu}, \quad \mu, \nu = 0, \dots, D-1. \quad (A.7)$$

Let us now address dualities. We begin with type IIB theory, which is invariant under S duality. We will have the axion field $C^{(0)}$ vanishing in our solutions; in that case under S-duality

$$\Phi \leftrightarrow -\Phi, \quad B_{\mu\nu}^{(2)} \leftrightarrow C_{\mu\nu}^{(2)}, \quad (\text{A.8})$$

while all other RR fields as well as the metric in the *Einstein frame* stay the same.

T duality, on the other hand, transforms IIA and IIB theories into each other. Let us assume that the space-like direction y is compactified on a circle of length $2\pi\sqrt{\alpha'}R$, then T duality transforms this coordinate into y' which now has identification $y' \sim y' + 2\pi\sqrt{\alpha'}/R$. The transformations of the NS fields are given by:

$$\begin{aligned} G'_{yy} &= \frac{1}{G_{yy}}, & e^{2\Phi'} &= \frac{e^{2\Phi}}{G_{yy}}, & G'_{\mu y} &= \frac{B_{\mu y}}{G_{yy}}, & B'_{\mu y} &= \frac{G_{\mu y}}{G_{yy}} \\ G'_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu y}G_{\nu y} - B_{\mu y}B_{\nu y}}{G_{yy}}, & B'_{\mu\nu} &= B_{\mu\nu} - \frac{B_{\mu y}G_{\nu y} - G_{\mu y}B_{\nu y}}{G_{yy}}, \end{aligned} \quad (\text{A.9})$$

while for the RR potentials we have:

$$C'^{(n)}_{\mu\dots\nu\alpha y} = C^{(n-1)}_{\mu\dots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\dots\nu]y}G_{|\alpha]y}}{G_{yy}}, \quad (\text{A.10})$$

$$C'^{(n)}_{\mu\dots\nu\alpha\beta} = C^{(n+1)}_{\mu\dots\nu\alpha\beta y} + nC^{(n-1)}_{[\mu\dots\nu\alpha}G_{\beta]y} + n(n-1)\frac{C^{(n-1)}_{[\mu\dots\nu]y}B_{|\alpha]y}G_{|\beta]y}}{G_{yy}}. \quad (\text{A.11})$$

We are interested in a special sequence of dualities (5.4). By looking at that equation, one can see that the RR fields should be transformed under T dualities only at one step: when we apply T6789 to D1 brane. Since we are starting from $C^{(1)}$ field pointing in x^5 direction, the only relevant transformation is (A.10), and we will end up with RR 6-field. Such field, however, does not enter the action for IIB theory directly, but it can be transformed into the two form $C^{(2)}$ by using electric-magnetic duality. Such duality simply means that the same configuration can be described by either p -form $C^{(p)}$ or $(8-p)$ -form $C^{(8-p)}$:

$$dC^{(p)} = *dC^{(8-p)}, \quad (\text{A.12})$$

where Hodge dual is taken with respect to string metric. In particular, the four form in type IIB theory is self-dual:

$$dC^{(4)} = *dC^{(4)}. \quad (\text{A.13})$$

In component form we have following relation between the field strengths:

$$G^{(p+1)\mu_1\dots\mu_{p+1}} = \frac{\epsilon^{\mu_1\dots\mu_{p+1}\nu_1\dots\nu_{9-p}}}{(9-p)!\sqrt{-G}} G^{(8-p)}_{\nu_1\dots\nu_{9-p}}, \quad (\text{A.14})$$

and we normalize ϵ by $\epsilon^{01\dots 9} = 1$.

While doing an S-duality we would like to have the metric remain $\eta_{\mu\nu}$ at infinity. This can be achieved by doing a rescaling of coordinates after the duality. Then the transformation rules for S-duality are

$$S : \quad g' = \frac{1}{g}, \quad R' = \frac{R}{\sqrt{g}}, \quad A' = \frac{A}{\sqrt{g}}, \quad a' = \frac{a}{\sqrt{g}}. \quad (\text{A.15})$$

For T duality in a direction y with identification $y \sim y + 2\pi\sqrt{\alpha'}R_y$ we have following transformation law:

$$T : \quad R'_y = \frac{1}{R_y}, \quad g' = \frac{g}{R_y}. \quad (\text{A.16})$$

B Boosts, dualities and metrics.

In this appendix we briefly describe some details of calculations omitted in section 3. In the process we will be able to write down explicitly the metrics for various 2-charge solutions (with angular momentum) that are encountered in the course of the duality maps – these solutions are written explicitly since they could be useful elsewhere. We will start from the solution (3.10) which describes fundamental string with momentum propagating along it, and apply the chain of dualities (5.4) to this solution. By applying S duality to (3.10) we get:

D1–P:

$$\begin{aligned} ds^2 &= -\sqrt{\frac{f_0}{f_A}}(dt^2 - dy^2) + \frac{2B^2}{\sqrt{f_0f_A}}(dt - \beta dy)^2 - \frac{4aAB}{\sqrt{f_0f_A}}\sin^2\theta d\phi(dt - \beta dy) \\ &+ \sqrt{f_0f_A}\left(\frac{dr^2}{r^2 + a^2} + d\theta^2\right) + \sqrt{\frac{f_A}{f_0}}(a^2 + r^2)\sin^2\theta d\phi^2 + \sqrt{\frac{f_A}{f_0}}r^2\cos^2\theta d\psi^2 \\ &+ \sqrt{\frac{f_A}{f_0}}\sum_{i=6}^9 dx^i dx^i \end{aligned} \quad (\text{B.1})$$

$$e^{2\Phi} = \frac{f_A}{f_0}, \quad C_{t\phi}^{(2)} = -\beta C_{y\phi}^{(2)} = \frac{2aAB\alpha\beta\sin^2\theta}{f_A}, \quad C_{ty}^{(2)} = -\frac{2A^2\alpha}{f_A}, \quad (\text{B.2})$$

Note that as the result of this S duality, the coordinates and parameters are rescaled according to rules represented by (5.5). For example, the relation between parameters A, B, a, R, V, g involved in the D1–P solution (B.1) and their counterparts for the FP solution, is given by:

$$g_{D1P} = \frac{1}{g_{FP}}, \quad A_{D1P} = \frac{A_{FP}}{\sqrt{g_{FP}}}, \quad B_{D1P} = \frac{B_{FP}}{\sqrt{g_{FP}}}, \quad a_{D1P} = \frac{a_{FP}}{\sqrt{g_{FP}}}, \quad (\text{B.3})$$

$$R_{D1P} = \frac{R_{FP}}{\sqrt{g_{FP}}}, \quad V_{D1P} = \frac{V_{FP}}{g_{FP}^2} \quad (\text{B.4})$$

It is the transformed parameters like A_{D1P} that appear in the metric (B.1) above, but for simplicity we omit the subscripts and write A in place of A_{D1P} . We will follow this convention in all the relations that follow.

Solution (B.1) describes the system of n_{D1} D1-branes with n_P units of momentum and angular momentum J :

$$n_{D1} = \frac{2A_{D1P}^2 V_{D1P} \alpha}{g_{D1P}}, \quad n_P = \frac{2B_{D1P}^2 V_{D1P} R_{D1P}^2 \beta}{g_{D1P}^2}, \quad (\text{B.5})$$

$$J = \frac{2a_{D1P} A_{D1P} B_{D1P} R_{D1P} V_{D1P}}{g_{D1P}^2} \quad (\text{B.6})$$

At the next step we apply a sequence of T dualities (T6789) to transform the D1 brane into the D5 brane. The result for the NS fields reads:

D5–P:

$$\begin{aligned} ds^2 &= -\sqrt{\frac{f_0}{f_A}}(dt^2 - dy^2) + \frac{2B^2}{\sqrt{f_0 f_A}}(dt - \beta dy)^2 - \frac{4aAB}{\sqrt{f_0 f_A}} \sin^2 \theta d\phi (dt - \beta dy) \\ &+ \sqrt{f_0 f_A} \left(\frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + \sqrt{\frac{f_A}{f_0}} (a^2 + r^2) \sin^2 \theta d\phi^2 + \sqrt{\frac{f_A}{f_0}} r^2 \cos^2 \theta d\psi^2 \\ &+ \sqrt{\frac{f_0}{f_A}} \sum_{i=6}^9 dx^i dx^i, \quad e^{2\Phi} = \frac{f_0}{f_A}. \end{aligned} \quad (\text{B.7})$$

The relation between parameters $A_{D5P}, B_{D5P}, a_{D5P}, R_{D5P}, V_{D5P}, g_{D5P}$ entering this solution and the parameters of (B.1) can be found using (5.5). Solution (B.7) describes the system of n_{D5} D5 branes with n_P units of momentum and angular momentum J :

$$n_{D5} = \frac{2A_{D5P}^2 \alpha}{g_{D5P}}, \quad n_P = \frac{2B_{D5P}^2 V_{D5P} R_{D5P}^2 \beta}{g_{D5P}^2}, \quad (\text{B.8})$$

$$J = \frac{2a_{D5P} A_{D5P} B_{D5P} R_{D5P} V_{D5P}}{g_{D5P}^2} \quad (\text{B.9})$$

Application of each of the four T dualities increases the rank of the Ramond–Ramond fields C by one. Thus the D5–P system has a RR six-form field (which is consistent with having a D5 brane) with following nontrivial components:

$$C_{t\phi 6789}^{(6)} = \beta C_{\phi y 6789}^{(6)} = \frac{2ABa\alpha\beta \sin^2 \theta}{f_A}, \quad C_{ty 6789}^{(6)} = -\frac{2\alpha A^2}{f_A}. \quad (\text{B.10})$$

However, in order to perform S duality on this solution, it is convenient to dualize above six-form field into the RR two-form field. This can be done using the general prescription of electric–magnetic duality (A.12). The nontrivial components of field strength for $C^{(6)}$

are:

$$\begin{aligned} G_{tr\phi 6789}^{(7)} &= -\beta G_{r\phi y 6789}^{(7)} = \frac{4rABa\alpha\beta \sin^2 \theta}{f_A^2} \\ G_{t\theta\phi 6789}^{(7)} &= -\beta G_{\theta\phi y 6789}^{(7)} = -\frac{2ABa\alpha\beta \sin 2\theta}{f_A^2}(r^2 + a^2 + 2A^2) \end{aligned} \quad (\text{B.11})$$

$$G_{try 6789}^{(7)} = -\frac{4\alpha A^2 r}{f_A^2}, \quad G_{t\theta y 6789}^{(7)} = \frac{2\alpha A^2 a^2 \sin 2\theta}{f_A^2} \quad (\text{B.12})$$

The field strength for the corresponding two-form is:

$$\begin{aligned} G_{\theta\phi\psi}^{(3)} &= -\frac{2\alpha A^2 r^2 (a^2 + r^2) \sin 2\theta}{f_0^2}, & G_{r\phi\psi}^{(3)} &= -\frac{\alpha a^2 A^2 r \sin^2 2\theta}{f_0^2} \\ G_{tr\psi}^{(3)} &= -\frac{4ABa\alpha r \cos^2 \theta}{f_0^2}, & G_{t\theta\psi}^{(3)} &= -\frac{4ABa\alpha r^2 \sin \theta \cos \theta}{f_0^2}, \\ G_{r\psi y}^{(3)} &= \frac{4ABa\alpha\beta r \cos^2 \theta}{f_0^2}, & G_{\theta\psi y}^{(3)} &= \frac{4ABa\alpha\beta r^2 \sin \theta \cos \theta}{f_0^2} \end{aligned} \quad (\text{B.13})$$

and the nontrivial components of the RR 2-form are:

$$\begin{aligned} C_{t\psi}^{(2)} &= -\frac{2aAB\alpha \cos^2 \theta}{f_0}, & C_{\phi\psi}^{(2)} &= \frac{2A^2\alpha(r^2 + a^2) \cos^2 \theta}{f_0}, \\ C_{\psi y}^{(2)} &= -\frac{2aAB\alpha\beta \cos^2 \theta}{f_0} \end{aligned} \quad (\text{B.14})$$

Application of S duality transforms this RR two form into the NS two-form field $B_{\mu\nu}$ and the metric and the dilaton of the transformed solution are given by:

NS5-P:

$$\begin{aligned} ds^2 &= -(dt^2 - dy^2) + \frac{2B^2}{f_0}(dt - \beta dy)^2 - \frac{4aAB}{f_0} \sin^2 \theta d\phi(dt - \beta dy) \\ &+ f_A \left(\frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + \frac{f_A}{f_0} (a^2 + r^2) \sin^2 \theta d\phi^2 + \frac{f_A}{f_0} r^2 \cos^2 \theta d\psi^2 \\ &+ \sum_{i=6}^9 dx^i dx^i, \quad e^{2\Phi} = \frac{f_A}{f_0}. \end{aligned} \quad (\text{B.15})$$

This solution has following charges:

$$n_{NS5} = 2A_{NS5P}^2 \alpha, \quad n_P = \frac{2B_{NS5P}^2 V_{NS5P} R_{NS5P}^2 \beta}{g_{NS5P}^2}, \quad (\text{B.16})$$

$$J = \frac{2a_{NS5P} A_{NS5P} B_{NS5P} R_{NS5P} V_{NS5P}}{g_{NS5P}^2} \quad (\text{B.17})$$

Performing two T dualities (T56) on the solution (B.15), we finally get the F1-NS5 system (5.6-5.7).

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